JOURNAL OF APPROXIMATION THEORY 75, 167–174 (1993)

The Expansion of a Holomorphic Function in a Series of Lamé Products

HANS VOLKMER

Department of Mathematical Sciences, University of Wisconsin–Milwaukee, P.O. Box 413, Milwaukee, Wisconsin 53201, U.S.A.

Communicated by Alphonse P. Magnus

Received July 30, 1991; accepted August 3, 1992

We show that functions of two complex variables which are symmetric and holomorphic on suitable domains can be expanded in locally uniform convergent series of products of Lamé polynomials. The result is based on a more general expansion theorem for holomorphic functions defined on a two-dimensional complex manifold which has been given in an earlier paper of the author.

1. INTRODUCTION

If we have been given a function of one complex variable which is holomorphic in the neigborhood of the interval $[-1, 1]$ then we can ask for the convergence properties of its Fourier expansion into a series of Legendre polynomials. The answer has been known since 1862, when K. Neumann [5] proved that the expansion converges locally uniformly in the interior domain of every ellipse with foci at -1 and 1 on which the function to be expanded is holomorphic. The proof can also be found in Schäfke [6, Chap. 8], Szegö [7, Thm. 9.1.1], Volkmer [10, Sect. 1], and Whittaker and Watson [11, Chap. XV].

There is an analogous question concerning the expansion of a holomorphic function of *two* variables into a series of Lamé products which we answer in the present paper. A Lamé product is a polynomial in two variables s, t of the form $E(s)\,E(t)$, where E is a Lamé polynomial. The system of Lamé products is orthogonal over the rectangle $[a, b] \times [b, c]$ with respect to a suitable weight, where $a < b < c$ are given. Hence we can again ask for the convergence properties of the corresponding Fourier series expansion of a function $g(s, t)$, which is now supposed to be holomorphic in the neighborhood of a rectangle. If, for a moment, we assume that there is a (perhaps very small) neighborhood of the rectangle

0021-9045/93 $5.00

on which the series converges uniformly to g then it follows that $g(s, t)$ is a symmetric function of its arguments in a neighborhood of the point (b, b). This is obvious because, by definition, the Lamé products are symmetric polynomials. We show that the converse is also true: if $g(s, t)$ is symmetric around (b, b) then its expansion into Lamé products is uniformly convergent to the sum g in a certain neighborhood of the rectangle. In fact, a more precise theorem which is analogous to Neumann's result on Legendre polynomials holds: every function $g(s, t)$ which is holomorphic and symmetric on the domain

$$G_\gamma = \left\{(s, t) \in \mathbb{C}^2 : \frac{|s-a|\,|t-a|}{|b-a|\,|c-a|} + \frac{|s-b|\,|t-b|}{|a-b|\,|c-b|} + \frac{|s-c|\,|t-c|}{|a-c|\,|b-c|} < \cosh 2\gamma\right\}, \tag{1.1}$$

where $\gamma > 0$, can be expanded into a series of Lamé products which converges locally uniformly on G_γ to the sum g. The above domains resemble the interior domains of ellipses with foci at a, b and half-axes $((b-a)/2) \cosh \delta$ and $((b-a)/2) \sinh \delta$ if we write these as

$$\left\{z \in \mathbb{C} : \frac{|z-a|}{|b-a|} + \frac{|z-b|}{|a-b|} < \cosh \delta\right\}.$$

The proof of the theorem depends on the well-known fact that Lamé products are the representations of spherical surface harmonics in sphero–conal coordinates. For the convenience of the reader we collect the definition of Lamé polynomials and their relation to sphero–conal coordinates and spherical surface harmonics on the two-dimensional sphere in Section 2. In Section 3 we consider the corresponding notions on a "complex sphere" which will enable us to apply a more general theorem on the expansion of a holomorphic function in spherical surface harmonics in order to obtain our theorem. We close with some remarks in Section 4.

Referring to the literature we mention that the expansion into a series of Lamé products was studied from the standpoint of real analysis by Dixon [1], Hilb [3], and others. A corresponding expansion of holomorphic functions of two complex variables was given by Volk [8]. Unfortunately, his paper is difficult to understand. In particular, his domains of convergence turn out to be incorrect if we try to give them the standard meaning.

2. THE EXPANSION OF SQUARE INTEGRABLE FUNCTIONS

Sphero–conal coordinates form an orthogonal system of coordinates on the sphere

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

We will work with the sphero–conal coordinates in the form used by Erdélyi, Magnus, Oberhettinger, and Tricomi [2, Sect. 15.1.2, (17)]. However it will be convenient to replace the constants a^2, b^2, c^2 appearing in that definition by $-a, -b, -c$. Throughout this paper a, b, c denote fixed real numbers such that $a < b < c$. We do not assume that these numbers are negative.

The sphero–conal coordinates (s, t) of a point $(x, y, z) \in S$ with positive x, y, z are defined as the solutions $u = s$, $u = t$ of the quadratic equation

$$\frac{x^2}{u-a} + \frac{y^2}{u-b} + \frac{z^2}{u-c} = 0, \tag{2.1}$$

where $a < s < b < t < c$. Conversely, (x, y, z) is given by

$$x^2 = \frac{(s-a)(t-a)}{(b-a)(c-a)},$$
$$y^2 = \frac{(s-b)(t-b)}{(a-b)(c-b)}, \qquad z^2 = \frac{(s-c)(t-c)}{(a-c)(b-c)}. \tag{2.2}$$

In this way the part $x > 0$, $y > 0$, $z > 0$ of the two-dimensional real analytic manifold S is mapped one-to-one and bianalytically onto the open rectangle $(s, t) \in R =]a, b[\times]b, c[$.

Let k be a given nonnegative integer and let $\theta_1, ..., \theta_k$ be real numbers different from a, b, c. Then consider the polynomial $f(x, y, z)$ defined by

$$f(x, y, z) = \prod_{i=1}^{k} \left(\frac{x^2}{\theta_i - a} + \frac{y^2}{\theta_i - b} + \frac{z^2}{\theta_i - c} \right), \tag{2.3}$$

which is homogeneous in x, y, z of degree $2k$. A direct calculation shows that this polynomial restricted to S has the form $E(s)\,E(t)$ in sphero–conal coordinates, where E is a polynomial of degree k with zeros at $\theta_1, ..., \theta_k$. The polynomial (2.3) is harmonic if and only if the corresponding polynomial $E(u)$ satisfies Lamé's differential equation

$$E'' + \frac{1}{2}\left(\frac{1}{u-a} + \frac{1}{u-b} + \frac{1}{u-c} \right) E'$$
$$- \frac{\lambda + n(n+1)u}{4(u-a)(u-b)(u-c)} E = 0, \tag{2.4}$$

with $n = 2k$ and a suitable value of the parameter λ. This can be shown as in Whittaker and Watson [11, Chap. XXIII], where the corresponding analysis is carried out in the case of elliptic coordinates. Another method of proof is to separate the Laplace equation on the sphere (the Laplace–Beltrami equation) in sphero–conal coordinates, which leads twice to Eq. (2.4) with $u = s$ and $u = t$, respectively. Then λ is the separation parameter. The polynomial solutions of (2.4) are the Lamé polynomials. For every even nonnegative integer n and every integer $m = 0, ..., n/2$ there exists precisely one (apart from a constant factor) Lamé polynomial of degree $n/2$ which has m zeros in the interval $]a, b[$ and the remaining $n/2 - m$ zeros in the interval $]b, c[$; see [11, Sect. 23.46]. We denote this Lamé polynomial by E_n^m.

Let $f_n^m(x, y, z)$ denote the spherical surface harmonic of the type (2.3) which has the representation $E_n^m(s)\, E_n^m(t)$ in sphero–conal coordinates. It is easy to show that the system of these spherical surface harmonics is orthogonal with respect to the usual inner product

$$\langle f_1, f_2 \rangle_S = \frac{1}{4\pi} \int_S f_1(p) \overline{f_2(p)}\, dp$$

for functions f_1, f_2 on the sphere. Transformed to sphero–conal coordinates, this inner product becomes

$$\langle g_1, g_2 \rangle_R = \frac{1}{2\pi} \int_a^b \int_b^c \frac{g_1(s, t)\, \overline{g_2(s, t)}\, (t-s)}{w(s)\, w(t)}\, dt\, ds, \tag{2.5}$$

where $w(u) = (|u-a|\, |u-b|\, |u-c|)^{1/2}$ and g_1, g_2 are the representations of f_1, f_2 in sphero–conal coordinates, respectively. We normalize the Lamé polynomials in such a way that, for every even nonnegative n, the system $f_n^m(x, y, z)$, $m = 0, ..., n/2$, forms an orthonormal basis in the $(n/2 + 1)$-dimensional space of spherical surface harmonics of degree n which are even with respect to each x, y, z. The orthogonal sum of these spaces is the Hilbert space of square integrable functions on the sphere S which are even with respect to each x, y, z. Summarizing we have the following result.

THEOREM 2.1. *The Lamé products*

$$(E_n^m \otimes E_n^m)(s, t) = E_n^m(s)\, E_n^m(t), \qquad n = 0, 2, 4, ..., m = 0, ..., n/2,$$

form a system of polynomials which is orthonormal and complete over the rectangle $R =]a, b[\times]b, c[$ *with respect to the inner product* $\langle \cdot, \cdot \rangle_R$ *given by* (2.5). *Every function g square integrable with respect to this inner product can be expanded into the* $\langle \cdot, \cdot \rangle_R$*-convergent series*

$$g(s, t) \sim \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \sum_{m=0}^{n/2} \langle g, E_n^m \otimes E_n^m \rangle_R\, E_n^m(s)\, E_n^m(t), \qquad (s, t) \in R. \tag{2.6}$$

There are similar theorems corresponding to the expansion of functions on the sphere having one of the remaining seven parities. Then the Lamé polynomials must be replaced by the corresponding types of Lamé functions. However, in the following we limit ourselves to the functions which are even with respect to each x, y, z. The other cases can be handled similarly. We remark that the above theorem is a particular case of a more general expansions theorem proved in multiparameter spectral theory; see [9, Sect. 6.10].

3. The Expansion of Holomorphic Functions

We now introduce sphero–conal coordinates on the two-dimensional complex manifold

$$T = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 + z^2 = 1\},$$

which was considered in [10, Sect. 3]. For a point $(x, y, z) \in T$ its sphero-conal coordinates are again defined as the solutions $u = s$, $u = t$ of Eq. (2.1). The numbers s, t are now complex, and it is no longer possible to distinguish s from t. Therefore we consider s, t as an unordered pair. The inverse transformation is again given by Eqs. (2.2). The correspondence $(x, y, z) \leftrightarrow (s, t)$ can be made precise in the following way. We call two points (x_1, y_1, z_1), $(x_2, y_2, z_2) \in T$ equivalent if $x_1^2 = x_2^2$, $y_1^2 = y_2^2$, $z_1^2 = z_2^2$ so that, in general, there are eight points in every equivalence class. Similarly, we call two points (s_1, t_1), $(s_2, t_2) \in \mathbb{C}^2$ equivalent if $(s_1, t_1) = (s_2, t_2)$ or $(s_1, t_1) = (t_2, s_2)$ so that, in general, there are two points in very equivalence class. Then we have a one-to-one correspondence between equivalence classes in T and in $\mathbb{C}^2$.

In [10] we considered subdomains

$$T_\gamma = \{(x, y, z) \in T : |x|^2 + |y|^2 + |z|^2 < \cosh 2\gamma\}$$

of T, where γ is a positive real number and $T_\infty = T$. By (2.2), we see that in sphero–conal coordinates these domains correspond to the sets G_γ defined in (1.1). We note that G_γ is a symmetric domain in $\mathbb{C}^2$, i.e., $(s, t) \in G_\gamma$ implies $(t, s) \in G_\gamma$. For $0 < \beta < \gamma \leqslant \infty$, we have

$$([b, c] \times [a, b]) \cup ([a, b] \times [b, c]) = \bigcap_{\alpha > 0} G_\alpha \subset G_\beta \subset G_\gamma \subset G_\infty = \mathbb{C}^2.$$

It is now clear from the remarks above that every function f on T_γ which is even with respect to each x, y, z corresponds to one and only one symmetric function g on G_γ by setting $f(x, y, z) = g(s, t)$. It is rather obvious that f is holomorphic if and only if g is holomorphic. We will use only the fact that holomorphy of g implies holomorphy of f so let us give the proof of this statement for the sake of completeness. Let g be holomorphic and symmetric on G_γ, and let (x_0, y_0, z_0) be a given point in T_γ with sphero–conal coordinates s_0, t_0. There are two cases. If $s_0 \neq t_0$ then the solutions s, t of (2.1) can be chosen as holomorphic functions of (x, y, z) in a neighborhood of (x_0, y_0, z_0), which implies that f is holomorphic there. If $s_0 = t_0$ then we consider the Taylor expansion of g at (s_0, s_0). Since g is symmetric this yields a sequence of symmetric polynomials $g_n(s, t)$ which converge uniformly to g in a neighborhood of (s_0, s_0). These polynomials can be written as polynomials in the elementary symmetric polynomials $s+t$ and st and these are holomorphic on T. Hence the functions $f_n(x, y, z) = g_n(s, t)$ are holomorphic on T and converge uniformly to f in a neighborhood of (x_0, y_0, z_0), which shows that f is holomorphic there.

Now consider the Lamé products $E_n^m(s)\, E_n^m(t)$ which are holomorphic and symmetric on $G_\infty = \mathbb{C}^2$. Hence the corresponding functions $f_n^m(x, y, z) = E_n^m(s)\, E_n^m(t)$ are holomorphic on T. They are therefore spherical surface harmonics of degree n on T in the sense of [10, Sect. 4]. By [10, Thm. 4.12], the Fourier expansion of every holomorphic function $f(x, y, z)$ on T_γ which is even with respect to each x, y, z into the system $f_n^m(x, y, z)$ converges locally uniformly on T_γ to the sum f. Transforming to sphero–conal coordinates we now obtain our main result.

THEOREM 3.1. *Let g be a holomorphic and symmetric function on the domain G_γ, where $0 < \gamma \leqslant \infty$. Then the expansion* (2.6) *of g into a series of Lamé products converges locally uniformly on G_γ to the sum g. More precisely, if $0 \leqslant \alpha < \beta < \gamma$ then we have the estimate*

$$|\langle g, E_n^m \otimes E_n^m \rangle_R\, E_n^m(s)\, E_n^m(t)| \leqslant M \frac{\pi}{2} (\coth \beta)(2n+1)^2\, e^{n(\alpha - \beta)} \tag{3.1}$$

for $(s, t) \in \bar{G}_\alpha$, where M is a bound for the absolute values of g on $\bar{G}_\beta$.

Since a function $g(s, t)$ which is holomorphic on a neighborhood of $[a, b] \times [b, c]$ and symmetric around (b, b) can be continued holomorphically onto a set G_γ with $\gamma > 0$ sufficiently small, we obtain the following corollary mentioned in the Introduction.

COROLLARY 3.2. *Let $g(s, t)$ be a function which is holomorphic on a neighborhood of $[a, b] \times [b, c]$. Then its Fourier expansion into a series of Lamé products is uniformly convergent on a (possibly smaller) neigborhood of $[a, b] \times [b, c]$ if and only if its Taylor series expansion at (b, b) is symmetric with respect to the variables s, t.*

4. Remarks on the Main Theorem

A theorem similar to that above was given by Volk [8, Thm. II, p. 227]. However, symmetry of the function to be expanded is never mentioned, and so this result cannot be true. We note that our variables s, t are related to the variables z, ζ used by Volk by $s = z^2$, $t = \zeta^2$. Volk works with domains in the complex plane which are bounded by curves of fourth order originally introduced by Lindemann [4]. D. Schmidt pointed out to me that these curves transformed back to our variables simply become ellipses with foci at the points b, c. The only possible interpretation of Volk's convergence domains is then that he uses cartesian products of interior domains of ellipses where we use the sets G_γ. However, simple examples which we do not include here show that we cannot replace the sets G_γ by sets of that form in Theorem 3.1. There are however relations between the sets G_γ and cartesian products of interior domains of ellipses which we give below. These relations will then be used to prove a corollary of Theorem 3.1 which is in the spirit of Volk's paper.

Consider the cut B_γ of the set G_γ belonging to $t = b$, i.e.,

$$B_\gamma = \{s \in \mathbb{C} : h(s, b) < \cosh 2\gamma\},$$

where

$$h(s, t) = \frac{|s-a|\,|t-a|}{|b-a|\,|c-a|} + \frac{|s-b|\,|t-b|}{|a-b|\,|c-b|} + \frac{|s-c|\,|t-c|}{|a-c|\,|b-c|}.$$

The set B_γ is the interior domain of the ellipse with foci at the points a, c and half-axes $((c-a)/2)\cosh 2\gamma$ and $((c-a)/2)\sinh 2\gamma$. We claim that

$$G_\gamma \subset B_\gamma \times B_\gamma \qquad \text{for all} \quad \gamma > 0. \tag{4.1}$$

The proof follows from the inequality $h(s, t) \geqslant h(s, b)$, which holds for all complex s, t. To prove the inequality we first remark that it is sufficient to consider real t. Then, for each fixed complex s, the function $h(s, \cdot)$ is linear on each of the four intervals $]-\infty, a]$, $[a, b]$, $[b, c]$, $[c, \infty]$, and it is easy to see that it is nonincreasing on the first two of them and nondecreasing on the last two of them.

Similarly, consider the cuts A_γ, C_γ of the set G_γ belonging to $t = a$ and $t = c$, respectively. The sets A_γ, C_γ are interior domains of ellipses with foci at b, c and a, b, respectively. The inequalities $h(s, t) \leqslant h(s, a)\,h(c, t)$ and $\cosh 2\sigma \cosh 2\tau \leqslant \cosh 2(\sigma + \tau)$ yield the inclusion

$$A_\sigma \times C_\tau \subset G_{\sigma+\tau} \qquad \text{for all} \quad \sigma, \tau > 0. \tag{4.2}$$

Combining (4.1) and (4.2) we obtain the following corollary of Theorem 3.1, in which no explicit reference is made to the sets G_γ.

THEOREM 4.1. *Let g be a holomorphic and symmetric function on* $B_\gamma \times B_\gamma$. *Then the expansion* (2.6) *of g into a series of Lamé products converges locally uniformly on* $A_\sigma \times C_\tau$ *whenever* $\sigma + \tau \leqslant \gamma$.

REFERENCES

1. A. C. DIXON, Expansions by means of Lamé functions, *Proc. London Math. Soc. (1)* **35** (1902), 162–197.
2. A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, AND F. G. TRICOMI, "Higher Transcendental Functions, III," McGraw–Hill, New York, 1955.
3. E. HILB, Die Reihenentwicklungen der Potentialtheorie, *Math. Ann.* **63** (1907), 38–53.
4. F. LINDEMANN, Entwicklung der Functionen einer komplexen Variablen nach Laméschen Funktionen und nach Zugeordneten der Kugelfunktionen, *Math. Ann.* **19** (1882), 323–386.
5. K. NEUMANN, Über die Entwicklung einer Funktion nach den Kugelfunktionen, Halle, 1862.
6. F. W. SCHÄFKE, "Einführung in die Theorie der speziellen Funktionen der mathematischen Physik," Springer-Verlag, Berlin/New York, 1963.
7. G. SZEGÖ, "Orthogonal Polynomials," Amer. Math. Soc., Providence, RI, 1939.
8. O. VOLK, Über die Entwicklung von Funktionen zweier komplexen Veränderlichen nach Laméschen Funktionen, *Math. Z.* **23** (1925), 224–237.
9. H. VOLKMER, "Multiparameter Eigenvalue Problems and Expansion Theorems," Lecture Notes in Mathematics, Vol. 1356, Springer-Verlag, Berlin/New York, 1988.
10. H. VOLKMER, The expansion of a holomorphic function in a Laplace series, *SIAM J. Math. Anal.* **21** (1990), 771–781.
11. E. T. WHITTAKER AND G. N. WATSON, "A Course of Modern Analysis," Cambridge Univ. Press, Cambridge, 1927.